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A simple nonlinear dissipative quantum evolution equation

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Abstract. In this paper we considered a nonlinear dissipative evolution equation that generalises the Schrödinger equation. In the corresponding evolution all the stationary states of the usual Schrödinger equation have a behaviour of semi-stable limit cycles, except the ground state which is stable. The model is applied to the spin-$\frac{1}{2}$ and to the damped harmonic oscillator. For the latter it is shown that the coherent states remain coherent and evolve as in the corresponding classical problem.

1. Introduction

As is well known, quantum theory has been very successful in its predictions of the atomic energy levels. Transitions between corresponding states have, however, only been described by means of a perturbation approach. This situation cannot be considered as being entirely satisfactory, one would prefer to be able to describe quantum transitions by means of an evolution equation for the state vectors ψ, elements of a Hilbert space $\mathcal{H}$. It is, however, well known that such an equation must be nonlinear, (Davies 1976, Andrade e Silva *et al* 1962). In this paper we consider a possible generalisation of the Schrödinger equation which, in addition to its simplicity, has several interesting properties.

We would like to emphasise that, regarding quantum mechanics from a fundamental point of view (Piron 1976), there are no *a priori* problems of compatibility between a nonlinear evolution equation and the basic principles of the usual Hilbert space formulation of quantum mechanics.

This problem of dissipative quantum evolution has already received quite a lot of attention (see for example the review articles of Messer (1979) and Hasse (1975) and the references therein). One of the first approaches is the one by Caldirola (1941) who uses an explicit time dependent Hamiltonian corresponding to a Lagrangian from which the exact classical damped evolution equation can be derived. However this Hamiltonian is, up to a particular time dependent canonical transformation, that of a conservative system (Lemos 1978 and Kernen 1958). A critical review of this approach has been made by Ray (1979). Albrecht (1975) proposed a class of simple friction potentials. Another approach based on the Hamilton–Jacobi equation is due to Stocker and Albrecht (1979). However, the best known approach to our problem is probably the one of Kostin (1975). Yasue (1976) has shown that Kostin's term arises naturally from the so-called stochastic quantisation procedure as applied to the classical friction

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term proportional to minus the velocity. However Kostin's term is quite complicated and it does not take into account the Hilbert space structure of quantum mechanics. Furthermore if one indeed has to expect that a quantum dissipative evolution exhibits some classical features in certain cases or limits, there may exist specific quantal aspects, without any classical counterparts. Remaud and Hernandez (1980) and Hasse (1978) have studied the motion of a wave packet with and without friction.

In the following we shall look at the problem from a new point of view. Most approaches to nonlinear dissipative Schrödinger like equations deal with nonlinear real potentials which are required to 'quantise' in some way the classical friction term. We shall not look at the problem in this way, and in fact our proposal does not even involve using a potential. Furthermore we shall not restrict ourselves to wave mechanics, i.e. to the special case $\mathscr{H} = \mathscr{L}^2(\mathbb{R}^3)$, but work in the general realm of quantum mechanics, with an arbitrary Hilbert space.

In the next section we present our model, give some properties of the corresponding evolution equation and discuss its solution. In § 3 we examine the cases of two- and three-level systems. In § 4 we shall use the Weyl transformation and discuss the formal limit $h \to 0$, in order to exhibit the classical counterpart of our evolution equation and to compare it with usual classical dissipative equations. Section 5 is devoted to the standard example, namely the damped harmonic oscillator.

2. The model, properties and solution

In this section we study the following simple nonlinear dissipative quantum evolution equation:

$$\dot{\psi} = -\mathrm{i}H\psi - \mathrm{i}\frac{ik}{\langle\psi|\psi\rangle}[\psi\psi^+; H]\psi \tag{1a}$$

$$= -\mathrm{i}H\psi + k(\langle H\rangle_\psi - H)\psi \tag{1b}$$

where H is the Hamiltonian of the system, and k is a positive real number.

The physical justification for the interest in this equation lies in its simplicity and in the properties mentioned below. Equation (1) also admits the possiblity of describing quantum phenomena which are not described by the Schrödinger equation. Indeed it may be that some phenomena are not described by the Schrödinger equation, especially when energy is exchanged in an irreversible way between two systems. For some applications of equation (1) to such cases see Gisin (1980) and Gisin and Piron (1981).

Let us list some important properties of equation (1):

(a) Equation (1) conserves the norm of ψ. From now on we shall assume that $|\psi| = 1$.

(b) If ψ is an eigenvector of H then equation (1) reduces to the usual Schrödinger equation.

(c) The evolution of the mean values of the (time independent) Hamiltonian reads: $(\mathrm{d}/\mathrm{d}t)\langle H\rangle_\psi = -2k(\Delta H)^2 \leqslant 0$ *and* $(\mathrm{d}/\mathrm{d}t)\langle H\rangle_\psi = 0$ iff ψ is an eigenvector of H.

Proof. From (1b) one gets:

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)\langle H\rangle_\psi &= 2k((\langle H\rangle_\psi)^2 - \langle H^2\rangle_\psi)\\ &= -2k(\Delta H)^2 \leqslant 0\end{aligned}$$

and

$$(\mathrm{d}/\mathrm{d}t)\langle H\rangle_\psi = 0 \Leftrightarrow \Delta H = 0$$

$$\Leftrightarrow H\psi = \lambda\psi \text{ for some } \lambda.$$

Notice that up to this point k could also have been a real positive valued function on $\mathscr{H}$, k_ψ.

(d) Equation (1) is invariant under a shift of the zero of energy.

(e) Equation (1) is covariant under time independent gauge transformations. More generally, it is covariant under any unitary transformation and invariant under the symmetry group of H.

(f) $\dot{\psi}$ is linear in H. Thus if H is larger, the system will follow the same evolution but correspondingly faster.

(g) If H is bounded, then the following function f

$$f: \mathscr{H} \to \mathscr{H}$$

$$\psi \mapsto -\mathrm{i}H\psi + k(\langle H\rangle - H)\psi$$

satisfies the Lifschitz condition and equation (1) has then a unique solution (see Berger 1977). The proof is immediate.

(h) The formal solution of equation (1) is:

$$\psi_t = \frac{\exp[-(\mathrm{i}+k)Ht]_{\psi_0}}{\langle\psi_0|\exp(-2kHt)|\psi_0\rangle^{1/2}} \tag{2}$$

where we have chosen $t = 0$ as the initial time.

In the case when the Hamiltonian H has a discrete spectrum this solution is given by:

$$\begin{aligned} &C_{i,\alpha}(t) = C_{i,\alpha}(0)\exp[-(\mathrm{i}+k)\lambda_i t]N(t)^{-1/2} \\ &N(t) = \sum_{i,\alpha} |C_{i,\alpha}(0)|^2 \exp[-2k\lambda_i t] \end{aligned} \tag{3}$$

where

$$\psi_t = \sum_{i,\alpha} C_{i,\alpha}(t)\varphi_{i,\alpha}$$

$$H\varphi_{i,\alpha} = \lambda_i\varphi_{i,\alpha}; \qquad \langle\varphi_{i,\alpha}|\varphi_{j,\beta}\rangle = \delta_{ij}\cdot\delta_{\alpha\beta}.$$

In words equations (3) say: In the case of a measurement of the energy of the system, the probability that the result is larger than the mean value of the Hamiltonian decreases as a function of time, whereas the probability that this result is less than this mean value increases.

A short analysis of equations (3) shows that almost all $|C_{i,\alpha}(t)|^2$ tend to zero when $t \to \infty$, except the ones that correspond to the lowest energy level for which the initial state has a non-vanishing coefficient. That is

$$\psi_t \xrightarrow{t\to\infty} (P_K\psi_0/|P_K\psi_0|)\exp(-\mathrm{i}\lambda_K t)$$

where P_K is the orthogonal projector onto the eigensubspace of H corresponding to λ_K,

$$\lambda_K = \min_i \{\lambda_i | P_i\psi_0 \neq 0\}.$$

We thus see that the usual stationary states, i.e. the eigenvectors of the Hamiltonian H, appear as asymptotic solutions of the evolution equation (1). Furthermore, these states are 'unstable below' in the following sense: if the initial state is an almost stationary state, namely an eigenvector of H to which we add an even small non-vanishing component on a lower level, then it will evolve to this lower energy asymptotic state. On the contrary these states are 'stable above'. Obviously the ground state is stable.

It may seem paradoxical to call an equation dissipative when it admits all eigenstates of H as stationary solutions. But it is no more paradoxical that a pyramid standing on its peak which, according to classical mechanics, is in an (unstable) equilibrium state.

(i) If H is bounded below, then ψ_t is the normalised projection of a unitary evolution which takes place on a larger Hilbert space $\mathscr{K}$:

$$\psi_t = \frac{P \exp(-\mathrm{i}Kt\varphi)}{\|P \exp(-\mathrm{i}Kt)\varphi\|}$$

where P is a projector defined on $\mathscr{K}$, $P\mathscr{K} = \mathscr{H}$, and K is a self-adjoint operator on $\mathscr{K}$ whose spectrum equals the whole real line.

Proof. If m is the lower bound of H, then $H' \equiv H + m \cdot \mathbb{1}$ is a positive operator and H and H' generate via equation (1) the same evolution. Now $\exp\{-(\mathrm{i}+k)H't\}$ is a contraction semigroup which, according to Sz-Nagy's theorem (Sz-Nagy and Foias 1970), can be dilated.

The problem of deducing the reduced deterministic dynamic of a subsystem will be studied in a forthcoming paper (Gisin 1981).

3. Example: systems with two or three levels

In the case where we have only two non-dengerate levels, as in a single spin-$\frac{1}{2}$ system, or if only two components of the initial state are non-zero, then, using equations (3), we obtain,

$$\langle H\rangle_t = \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2}\tanh(k(\lambda_2-\lambda_1)t + \ln|c_1(0)/c_2(0)|)$$

where $\lambda_1 < \lambda_2$ denote the values of the energy of the two levels. The graph of $\langle H\rangle_t$ is shown in figure 1.

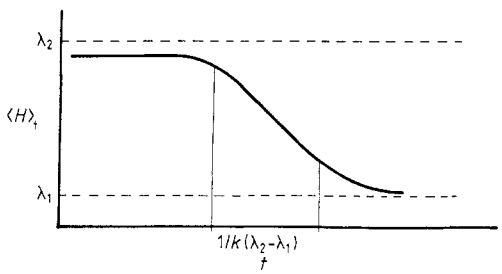

Figure 1. Decreasing of the energy as a function of time for a two-level system.

If exactly three components of the initial state are non-vanishing then the graph of $\langle H\rangle_t$ depends strongly on the exact initial conditions. For example if $|C_3(0)| \gg |C_2(0)| \gg |C_1(0)|$, then it has the form shown in figure 2. Whereas for $|C_3(0)| \gg |C_1(0)| \gg |C_2(0)|$, it is that of figure 3.

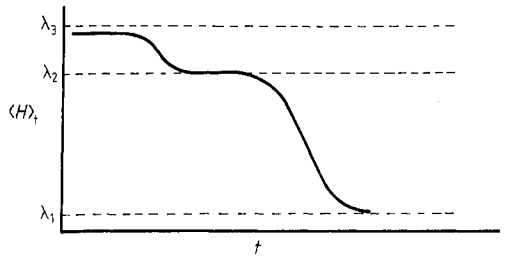

Figure 2. Decreasing of the enery as a function of time for a three-level system with initial condition $|C_3| \gg |C_2| \gg |C_1|$.

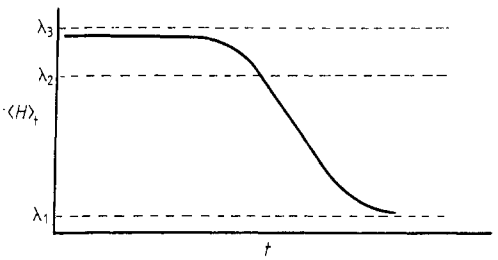

Figure 3. Decreasing of the energy as a function of time for a three-level system with initial condition $|C_3| \gg |C_1| \gg |C_2|$.

4. Weyl transformation and classical limit

In this section we apply a Weyl transformation to equation (1). Then we take the formal limit $h \to 0$.

First we have to rewrite our equation (1) as an evolution equation for the projector $\rho = |\psi\rangle\langle\psi|$ and introduce the Planck constant divided by 2π, $\hbar$:

$$\dot{\rho} = -(\mathrm{i}/\hbar)[H, \rho]_- - \frac{k}{\hbar^2}[[H, \rho], \rho]. \tag{4}$$

Using standard results (de Groot 1975) we obtain the classical *analogue* of equation (4), (in one space dimension):

$$\partial_t \rho(p, q, t) = \{h, \rho\} + k\{\{h, \rho\}, \rho\} \tag{5}$$

where $\rho(p, q, t)$ now denotes the distribution function over the classical phase space, $h(p, q, t)$ is the Weyl transformation of the Hamiltonian H, and $\{\ ,\ \}$ denotes the Poisson bracket.

If (p_m, q_m) is a local maxima of $\rho(p, q, t_0)$, and if $h(p, q) = p^2/2m + V(q)$, then equation (5) implies:

$$\dot{q}(p_m, q_m, t_0) = \partial_p h(p_m, q_m) - k\mu_{pq}\partial q V(q_m) \tag{6a}$$

$$\dot{p}(p_m, q_m, t_0) = -\partial_q h(p_m, q_m) - k\mu_q \frac{p_m}{m} \tag{6b}$$

where $\mu_j = -\partial_j^2 \rho(p_m, q_m, t_0)$, $j = p, q$.

Because of the limit $h \to 0$ equation (5) cannot of course exhibit all the features of our quantum evolution. Equation (6a) however, implies that $q(p_m, q_m, t_0)$ tends to a local minimum of the potential $V(q)$, and equation (6b) contains the well known friction term proportional to $-p$. Notice that the second term on the right-hand side of equation (6a) can be removed by an appropriate gauge transformation.

5. Damped harmonic oscillator

The damped harmonic oscillator is the standard example of a dissipative system. Many authors have tried to quantise it, Dekker (1979), Huguenin (1978) among others. From the preceding sections we already know that our model will exhibit different behaviour from that of the classical damped oscillator: the system will evolve to the lowest energy eigenstate of the usual quantum oscillator for which the initial state has a non-vanishing coefficient. An interesting question is to ask what happens to the coherent states, also called quasi-classical states. As any coherent state has a non-vanishing component in the ground state we know that the system will tend asymptotically to this state. But the surprising fact is that coherent states remain coherent during their evolution under equation (1), and follow the path of a classical damped oscillator. The quasi-classical states of the damped quantum oscillator thus exhibit classical features, although we have introduced *a priori* no classical friction term.

With obvious notation we have:

$$H = p^2/2m + m\omega^2 q^2/2$$

$$\hat{p} = p/\sqrt{m\omega\hbar}, \qquad \hat{q} = (m\omega/\hbar)^{1/2} q$$

$$a = (\hat{q} + i\hat{p})/\sqrt{2}$$

$$H = \hbar\omega\hat{H}, \qquad \hat{H} = (\hat{p}^2 + \hat{q}^2)/2 = a^+ a + \tfrac{1}{2}.$$

Let us denote by $|\alpha\rangle$ the coherent states, i.e. the eigenstates of the 'annihilation' operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \text{with } \alpha \in \mathbb{C}.$$

Then the solution of equation (1) with the initial condition $\psi_0 = |\alpha_0\rangle$ is:

$$\psi_t = |\alpha_t\rangle$$

where α_t is a complex-valued function of time satisfying the following equation:

$$\dot{\alpha}_t = -i\omega\alpha_t - k\omega\alpha_t.$$

From this last equation one deduces that:

$$\dot{\langle p\rangle}_{\alpha_t} = -m\omega^2\langle q\rangle_{\alpha_t} - k\omega\langle p\rangle_{\alpha_t} \tag{7a}$$

$$\dot{\langle q\rangle}_{\alpha_t} = m^{-1}\langle p\rangle_{\alpha_t} - k\omega\langle q\rangle_{\alpha_t} \tag{7b}$$

and

$$\langle q\rangle^{\cdot\cdot}_{\alpha_t} + \omega_0^2 \langle q\rangle_{\alpha_t} + 2\lambda \langle q\rangle^{\cdot}_{\alpha_t} = 0 \tag{8}$$

where

$$\omega_0^2 = \omega^2(1+k^2) \tag{9a}$$

$$\lambda = k\omega. \tag{9b}$$

Equation (8) is the usual evolution equation of the classical damped oscillator. Equations (9) imply that $\omega_0 > \lambda$, i.e. the damping is below critical. In this case an appropriate gauge transformation can be used to absorb the second term on the right-hand side of equation (7*b*), so that equations (7) are equivalent to the usual classical equations.

6. Conclusions

We would like to emphasise that we did not start with a classical friction term, and then try to quantise it, as has often been done. On the contrary we started within a quantum theoretical context, describing the states of the physical system by normed vectors in a Hilbert space. Thus the quite simple nonlinear equation (1) is a fully quantal evolution law, compatible with any Hilbert space. For example compare figures 4 and 5 which show two solutions of our model, one for a damped spin-$\frac{1}{2}$, the other for the coherent state of the damped harmonic oscillator. In § 2 we gave its complete solution, and the qualitative analysis we made in §§ 2 and 3 shows some nice properties of this model. Namely the fact that the eigenvectors of the Hamiltonian H appear as a kind of limit cycle, and that only the ground state is stable. Sections 4 and 5 show that the quasi-classical states of the harmonic oscillator, and any state in the limit $h \to 0$, exhibit classical features.

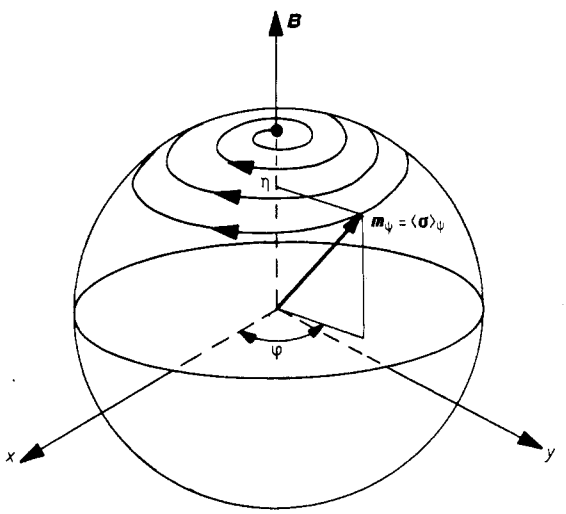

Figure 4. Solution of equation (1) for the damped spin-$\frac{1}{2}$ system.

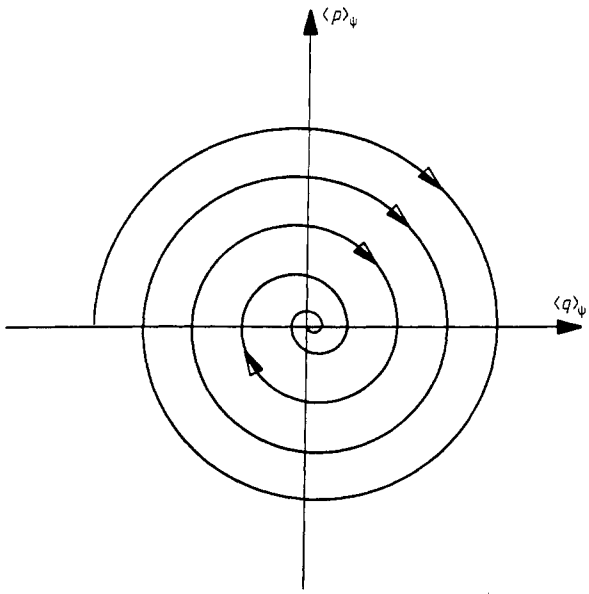

Figure 5. Solution of equation (1) for the coherent state of the damped harmonic oscillator.

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